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2007 J. Phys. A: Math. Theor. 40 14803

(http://iopscience.iop.org/1751-8121/40/49/013)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 14803–14810 doi:10.1088/1751-8113/40/49/013

On the single-exponential closed form of the product of two exponential operators

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Received 12 August 2007, in final form 18 October 2007
Published 21 November 2007
Online at stacks.iop.org/JPhysA/40/14803

Abstract

In the frame of a vectorial Pauli algebraic approach it is shown that the product of two exponentials of any two-by-two linear operators can be put in a single-exponential closed form. As a first application, a compact vectorial expression for the characteristics (angle and axis) of the product of two $\mathbf{R}^3$ rotations is established. The same mathematics can be used in a large diversity of problems of the whole class of two-states physical systems. An exemplification in the field of polarization optics is given.

PACS numbers: 02.30.Tb, 42.25.Ja

1. Introduction

A quite general problem, often encountered in linear algebra and its applications in physics is that of expressing the product of two exponential operators in the form of a unique exponential operator:

$$\mathrm{e}^A\,\mathrm{e}^B = \mathrm{e}^C, \tag{1}$$

in other words of determining $C = \ln \mathrm{e}^A\,\mathrm{e}^B$, where A and B are some general operators (of the same type).

It is well known (e.g. [1]) that if the two operators at the exponents commute

$$[A, B] = 0, \tag{2}$$

the two exponentials in equation (1) commute too and

$$\mathrm{e}^A\,\mathrm{e}^B = \mathrm{e}^B\,\mathrm{e}^A = \mathrm{e}^{A+B}. \tag{3}$$

A problem which naturally arises here is what is the difference between $\mathrm{e}^C = \mathrm{e}^A\,\mathrm{e}^B$ and e^{A+B} when A and B do not commute. Evidently, this difference depends on the commutatorial properties of the pair of operators A, B.

A particular solution to this problem is when both the operators commute with their commutator,

$$[A, [A, B]] = [B, [A, B]] = 0. \tag{4}$$

In this case,

$$\mathrm{e}^{A}\,\mathrm{e}^{B} = \mathrm{e}^{A+B}\,\mathrm{e}^{\frac{1}{2}[A,B]} = \mathrm{e}^{A+B+\frac{1}{2}[A,B]}. \tag{5}$$

The general solution of the problem is given by the Baker–Campbell–Hausdorff (BCH) formula, which expresses $\mathrm{e}^{A}\,\mathrm{e}^{B}$ as an exponential of an infinite series of commutator terms [2],

$$\mathrm{e}^{A}\,\mathrm{e}^{B} = \mathrm{e}^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\frac{1}{12}[[A,B],B]+\cdots}. \tag{6}$$

The series cannot generally be summed, explicitly, but the recursion scheme which gives the terms of the series may, in principle, be carried out to arbitrarily high order. Weiss and Maradudin [3] have calculated the series out to the fifth order, Eriksen [4] up to the sixth order, while Richtmyer and Greenspan [5] have calculated it out to the 512 order by computer and published it to the tenth order.

The, generally very complicated, exponential series of commutators are greatly reduced if the operators A and B pertain to a low-dimensional Lie algebra, and there are some particular cases when $\mathrm{e}^{A}\,\mathrm{e}^{B}$ can be written as a closed-form single exponential [2, 6].

In the case of the linear operators defined on a unitary space of dimension two over the field of complex numbers, i.e. in the case of the GL(2, C) operators, an exact solution to this problem exists. However for 2×2 matrices the BCH formula does not always converge. Since our concern is with applications, criteria for convergence will not be discussed in this paper.

The core theoretical result of the paper, equations (24) and (25), has been known for a long time. It is stated, without demonstration and in another parameterization, in the appendix of the paper [7] by Dragt and Finn. I give here a demonstration of this result in a parameterization which leads to a form more symmetric and, I think, better suited to a certain kind of physical applications, namely the analysis of 'two-state' or 'two-beam' systems. A prospect for the application of this theoretical result is opened in sections 5 and 6 of the paper.

We shall adopt here a vectorial Pauli algebraic approach, and it is expectable that it will transpose the problem in some equations referring to the Pauli axes [8] of the involved operators.

2. The commutator $[\mathrm{e}^{A}, \mathrm{e}^{B}]$ in function of the commutator $[A, B]$

Any operator of class GL(2, C) may be expanded in the form

$$A = a_0\sigma_0 + \mathbf{a}\cdot\boldsymbol{\sigma}, \tag{7}$$

where $(\sigma_0, \boldsymbol{\sigma})$ is the Stokes vector of the Pauli matrices, a_0 is a (generally complex) scalar and $\mathbf{a}$ is, generally, a complex vector in $\mathbf{C}^3$ (see [7] and further citation herein). If we denote by $\mathbf{m}$ the unit vector corresponding to $\mathbf{a}$ (we have denominated this unit vector as the Pauli axis of the operator [7], and we shall call $\mathbf{a}$ the Pauli vector of the operator), and by μ the modulus of this vector, equation (7) takes the form

$$A = a_0\sigma_0 + \mu\mathbf{m}\cdot\boldsymbol{\sigma}. \tag{8}$$

Similarly, write B in the form

$$B = b_0\sigma_0 + \nu\mathbf{n}\cdot\boldsymbol{\sigma}, \tag{9}$$

Then we have

$$e^A = e^{a_0\sigma_0}\, e^{\mu\mathbf{m}\cdot\boldsymbol{\sigma}} = e^{a_0\sigma_0}(\sigma_0 \cosh\mu + \mathbf{m}\cdot\boldsymbol{\sigma}\sinh\mu), \tag{10}$$

$$e^B = e^{b_0\sigma_0}\, e^{\nu\mathbf{n}\cdot\boldsymbol{\sigma}} = e^{b_0\sigma_0}(\sigma_0 \cosh\nu + \mathbf{n}\cdot\boldsymbol{\sigma}\sinh\nu) \tag{11}$$

and

$$\begin{aligned} e^A\, e^B &= e^{(a_0+b_0)\sigma_0}(\sigma_0\cosh\mu + \mathbf{m}\cdot\boldsymbol{\sigma}\sinh\mu)(\sigma_0\cosh\nu + \mathbf{n}\cdot\boldsymbol{\sigma}\sinh\nu) \\ &= e^{(a_0+b_0)\sigma_0}\,[(\cosh\mu\cosh\nu + \mathbf{m}\cdot\mathbf{n}\sinh\mu\sinh\nu)\,\sigma_0 \\ &\quad + (\mathbf{m}\sinh\mu\cosh\nu + \mathbf{n}\cosh\mu\sinh\nu)\cdot\boldsymbol{\sigma} + \mathrm{i}(\mathbf{m}\times\mathbf{n})\cdot\boldsymbol{\sigma}\sinh\mu\sinh\nu]\,, \end{aligned} \tag{12}$$

where we have used Dirac's formula

$$(\mathbf{m}\cdot\boldsymbol{\sigma})(\mathbf{n}\cdot\boldsymbol{\sigma}) = \mathbf{m}\cdot\mathbf{n} + \mathrm{i}(\mathbf{m}\times\mathbf{n})\cdot\boldsymbol{\sigma}. \tag{13}$$

Herefrom, it is straightforward that

$$[e^A, e^B] = 2\mathrm{i}(\mathbf{m}\times\mathbf{n})\cdot\boldsymbol{\sigma}\, e^{(a_0+b_0)\sigma_0}\sinh\mu\sinh\nu, \tag{14}$$

with $\mathbf{m}$ and $\mathbf{n}$, we stress once again, generally complex unit vectors.

On the other hand, from equations (8) and (9) we may calculate the commutator of the operators A and B:

$$\begin{aligned} [A, B] &= (a_0\sigma_0 + \mu\mathbf{m}\cdot\boldsymbol{\sigma})(b_0\sigma_0 + \nu\mathbf{n}\cdot\boldsymbol{\sigma}) - (b_0\sigma_0 + \nu\mathbf{n}\cdot\boldsymbol{\sigma})(a_0\sigma_0 + \mu\mathbf{m}\cdot\boldsymbol{\sigma}) \\ &= 2\mathrm{i}\mu\nu(\mathbf{m}\times\mathbf{n})\cdot\boldsymbol{\sigma}. \end{aligned} \tag{15}$$

Hence, quite generally, for the 2×2 operators, we have

$$[e^A, e^B] = \frac{\sinh\mu\sinh\nu}{\mu\nu}\, e^{(a_0+b_0)\sigma_0}[A, B], \tag{16}$$

where μ and ν are the moduli of the Pauli vectors of the operators A and B, respectively.

From equations (14) and (15) it is evident that, aside from some trivial conditions, the operators e^A and e^B as well as A and B commute when their Pauli (generally complex) axes are collinear (generally in $\mathbf{C}^3$):

$$\mathbf{m}\times\mathbf{n} = 0 \quad\Leftrightarrow\quad \mathbf{m}\equiv\mathbf{n} \quad\Leftrightarrow\quad \mathbf{m} = \lambda\mathbf{n}, \tag{17}$$

where $\equiv$ stands here for 'congruent' or 'equivalent' (in the sense of colinearity) and λ is generally a complex number of modulus 1.

This is the Pauli algebraic condition for being allowed to write equation (3).

Even if in the applications to which we shall refer, the unit vectors $\mathbf{m}$, $\mathbf{n}$ are real, these results are valid also for complex vectors.

Finally, from equation (16) we get

$$e^A\, e^B = e^B e^A + \frac{\sinh\mu\sinh\nu}{\mu\nu} e^{(a_0+b_0)\sigma_0}[A, B]. \tag{18}$$

Now we are able to look for the concrete Pauli algebraic form of equation (1).

3. The closed-form single exponential of $e^A\, e^B$

If we write the Pauli expansion of C as

$$C = c_0\sigma_0 + \gamma\mathbf{c}\cdot\boldsymbol{\sigma}, \tag{19}$$

with $\mathbf{c}$ being a unit vector, the Pauli axis of C, then we have

$$e^A\, e^B = e^C = e^{c_0\sigma_0}\, e^{\gamma\mathbf{c}\cdot\boldsymbol{\sigma}} = e^{c_0\sigma_0}(\sigma_0\cosh\gamma + \mathbf{c}\cdot\boldsymbol{\sigma}\sinh\gamma), \tag{20}$$

and identifying between equations (20) and (12) we get for the Pauli elements of the operator C:

$$c_0 = a_0 + b_0, \tag{21}$$

$$\cosh\gamma = \cosh\mu\cosh\nu + \mathbf{m}\cdot\mathbf{n}\sinh\mu\sinh\nu, \tag{22}$$

$$\mathbf{c}\sinh\gamma = \mathbf{m}\sinh\mu\cosh\nu + \mathrm{i}(\mathbf{m}\times\mathbf{n})\sinh\mu\sinh\nu + \mathbf{n}\cosh\mu\sinh\nu. \tag{23}$$

In the general case ($\mathbf{m} \neq \mathbf{n}$) the formulae for $\mathbf{c}$ and γ are not very simple but, as we shall see, they have a high degree of symmetry and of mathematical expressivity. With equations (22) and (23) we get

$$\gamma = \operatorname{arc\,sinh}\sqrt{\cosh^2\mu\cosh^2\nu + (1/2)\mathbf{m}\cdot\mathbf{n}\sinh 2\mu\sinh 2\nu + (\mathbf{m}\cdot\mathbf{n})^2\sinh^2\mu\sinh^2\nu - 1}, \tag{24}$$

$$\mathbf{c} = \frac{\mathbf{m}\sinh\mu\cosh\nu + \mathrm{i}(\mathbf{m}\times\mathbf{n})\sinh\mu\sinh\nu + \mathbf{n}\cosh\mu\sinh\nu}{\sqrt{\cosh^2\mu\cosh^2\nu + (1/2)\mathbf{m}\cdot\mathbf{n}\sinh 2\mu\sinh 2\nu + (\mathbf{m}\cdot\mathbf{n})^2\sinh^2\mu\sinh^2\nu - 1}}. \tag{25}$$

Equations (24) and (25) determine exactly the Pauli elements γ, $\mathbf{c}$ of the operator C as functions of the Pauli elements μ, $\mathbf{m}$ and ν, $\mathbf{n}$ of the operators A and B. This way the solution C of equation (1) is determined. The product $\mathrm{e}^A\,\mathrm{e}^B$ can be expressed as a closed-form single exponential $\mathrm{e}^C = \mathrm{e}^{\gamma\mathbf{c}\cdot\boldsymbol{\sigma}}$, with γ and $\mathbf{c}$ determined by equations (24) and (25), respectively.

It is easy to verify that

$$\begin{aligned}&\sqrt{\cosh^2\mu\cosh^2\nu + (1/2)\mathbf{m}\cdot\mathbf{n}\sinh 2\mu\sinh 2\nu + (\mathbf{m}\cdot\mathbf{n})^2\sinh^2\mu\sinh^2\nu - 1}\\ &\qquad = \|\mathbf{m}\sinh\mu\cosh\nu + \mathrm{i}(\mathbf{m}\times\mathbf{n})\sinh\mu\sinh\nu + \mathbf{n}\cosh\mu\sinh\nu\|.\end{aligned} \tag{26}$$

Thus $\mathbf{c}$ is a unit vector as we have conceived it in equation (19).

On this basis, the expressions of the parameters γ, $\mathbf{c}$ of the operator C take a remarkably symmetric form as function of the corresponding parameters of A and B:

$$\gamma = \operatorname{arc\,sinh}\|\mathbf{m}\sinh\mu\cosh\nu + \mathrm{i}(\mathbf{m}\times\mathbf{n})\sinh\mu\sinh\nu + \mathbf{n}\cosh\mu\sinh\nu\|, \tag{27}$$

$$\mathbf{c} = \frac{\mathbf{m}\sinh\mu\cosh\nu + \mathrm{i}(\mathbf{m}\times\mathbf{n})\sinh\mu\sinh\nu + \mathbf{n}\cosh\mu\sinh\nu}{\|\mathbf{m}\sinh\mu\cosh\nu + \mathrm{i}(\mathbf{m}\times\mathbf{n})\sinh\mu\sinh\nu + \mathbf{n}\cosh\mu\sinh\nu\|}. \tag{28}$$

4. Application: the product of two rotations

Let us consider now, as an application of the above-established general formulae, the classical problem of the product of two rotations in $\mathbf{R}^3$. We shall multiply two rotations (given by the corresponding unitary operators):

- one of the angle δ_1 around the axis $\mathbf{n}_1$:

$$U_{\mathbf{n}_1}(\delta_1) = \mathrm{e}^{-\mathrm{i}\frac{\delta_1}{2}\mathbf{n}_1\cdot\boldsymbol{\sigma}}, \tag{29}$$

- the second of angle δ_2 around the axis $\mathbf{n}_2$:

$$U_{\mathbf{n}_2}(\delta_2) = \mathrm{e}^{-\mathrm{i}\frac{\delta_2}{2}\mathbf{n}_2\cdot\boldsymbol{\sigma}}. \tag{30}$$

The result of these two rotations is another rotation whose corresponding unitary operator is

$$U_n(\delta) = U_{\mathbf{n}_2}(\delta_2)U_{\mathbf{n}_1}(\delta_1) = \mathrm{e}^{-\mathrm{i}\frac{\delta}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}, \tag{31}$$

and whose axis $\mathbf{n}$ and angle δ have to be determined.

Let us see what become our general formulae in this particular case.

The linear Pauli algebraic expansion of (the unitary operator corresponding to) the resultant rotation is given by equation (12):

$$\begin{aligned} U_{\mathbf{n}}(\delta) = &\left(\cos\frac{\delta_2}{2}\cos\frac{\delta_1}{2} - \mathbf{n}_2\cdot\mathbf{n}_1\sin\frac{\delta_2}{2}\sin\frac{\delta_1}{2}\right)\sigma_0 \\ &- \mathrm{i}\left(\mathbf{n}_2\sin\frac{\delta_2}{2}\cos\frac{\delta_1}{2} + \mathbf{n}_2\times\mathbf{n}_1\sin\frac{\delta_2}{2}\sin\frac{\delta_1}{2} + \mathbf{n}_1\cos\frac{\delta_2}{2}\sin\frac{\delta_1}{2}\right)\cdot\boldsymbol{\sigma}, \end{aligned} \tag{32}$$

In what concerns the commutator of the two rotations, it is given by equation (14):

$$\left[U_{\mathbf{n}_2}(\delta_2), U_{\mathbf{n}_1}(\delta_1)\right] = -2\mathrm{i}(\mathbf{n}_2\times\mathbf{n}_1)\cdot\boldsymbol{\sigma}\sin\frac{\delta_2}{2}\sin\frac{\delta_1}{2}. \tag{33}$$

The characteristic elements of the resulting rotation $(\mathbf{n}, \delta)$ in function of those of the composed rotations $(\mathbf{n}_1, \delta_1)$, $(\mathbf{n}_2, \delta_2)$ are given by equations (22) and (23) or (24) and (25), or (27) and (28).

From equation (22) we get

$$\delta = 2\arccos\left(\cos\frac{\delta_2}{2}\cos\frac{\delta_1}{2} - \mathbf{n}_2\cdot\mathbf{n}_1\sin\frac{\delta_2}{2}\sin\frac{\delta_1}{2}\right), \tag{34}$$

and from equation (28),

$$\mathbf{n} = \frac{\mathbf{n}_2\sin\frac{\delta_2}{2}\cos\frac{\delta_1}{2} + (\mathbf{n}_2\times\mathbf{n}_1)\sin\frac{\delta_2}{2}\sin\frac{\delta_1}{2} + \mathbf{n}_1\cos\frac{\delta_2}{2}\sin\frac{\delta_1}{2}}{\left\|\mathbf{n}_2\sin\frac{\delta_2}{2}\cos\frac{\delta_1}{2} + (\mathbf{n}_2\times\mathbf{n}_1)\sin\frac{\delta_2}{2}\sin\frac{\delta_1}{2} + \mathbf{n}_1\cos\frac{\delta_2}{2}\sin\frac{\delta_1}{2}\right\|}. \tag{35}$$

5. Application: orthogonal and non-orthogonal polarization devices

Let us put this application in some concrete physical terms. We shall consider a problem widely encountered in the theory of the polarization devices. Generally, a polarization device or a polarization arrangement is composed by a series of elementary ('canonical') devices: homogeneous polarizers, retarders, rotators [9–11]. The 2×2 operators (in the Jones formalism) of these canonical devices are normal operators: their eigenvectors are orthogonal. Therefore and in this sense, the canonical devices are called also orthogonal devices [12].

The canonical retarders are represented by unitary operators.

Apart from an unessential phase factor, the unitary operators have the Pauli algebraic form [8],

$$U_{\mathbf{n}}(\delta) = \mathrm{e}^{-\mathrm{i}\frac{\delta}{2}\mathbf{n}\cdot\boldsymbol{\sigma}} = \sigma_0\cos\frac{\delta}{2} - \mathrm{i}\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\delta}{2}. \tag{36}$$

The canonical, homogeneous polarizers are represented by Hermitian operators. Again we can let aside an unessential real factor, corresponding to the isotropic transmission of the polarizer. The essential part of the operator of a homogeneous polarizer is constituted by a unimodular Hermitian operator.

The Pauli algebraic expansion of a unimodular Hermitian (boost, squeeze) operator is [8]

$$H_{\mathbf{m}}(\eta) = \mathrm{e}^{\frac{\eta}{2}\mathbf{m}\cdot\boldsymbol{\sigma}} = \sigma_0\cosh\frac{\eta}{2} + \mathbf{m}\cdot\boldsymbol{\sigma}\sinh\frac{\eta}{2}. \tag{37}$$

In the representations (36) and (37) of the unitary and the unimodular Hermitian operators, both the unit vectors $\mathbf{n}$ and $\mathbf{m}$ are real. They are the Poincaré axes of the two operators. The squeeze may be conceived also as an imaginary rotation [p 93, 13] of $\delta \in \mathrm{i}\mathbf{R}$ around a real axis, or as a real rotation around an imaginary axis $\mathbf{n} \in \mathrm{i}\mathbf{R}^3$ [p 189, 14].

Both the operators (36) and (37) are evidently normal operators. They pertain to the special linear group, the 'unimodular group' [p 297, 1] SL(2, C): the unitary operators are implicitly unimodular, the Hermitian operators (37) by the restriction we have imposed.

A series of orthogonal devices (each characterized by a normal operator) gives rise, generally to a non-orthogonal device (non-orthogonal eigenvectors, non-normal operator) [15–17]. But any operator (normal or non-normal) may be expressed, by the polar decomposition, as the product of a Hermitian operator and a unitary operator [15, 18, 19]. Particularly by means of a various pairs of operators (36), (37) we can build up the whole group SL(2, C):

$$L_{2,C} = H_{\mathbf{m}}(\eta) U_{\mathbf{n}}(\delta) = \mathrm{e}^{\frac{\eta}{2}\mathbf{m}\cdot\boldsymbol{\sigma}}\, \mathrm{e}^{-\mathrm{i}\frac{\delta}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}. \tag{38}$$

Referring to our problem, equation (38) means that any composed polarization device can be reduced to (can be conceived as) a succession of a polarizer and a retarder (both generally elliptic). In this way, its birefringent and dichroic properties can be separated.

We have used here the right-hand polar decomposition. A left-hand decomposition may be equally used. In general, these two decompositions are different, i.e. the two factors in equation (38) do not commute.

Let us first to expand the general Pauli expression (38) of a SL(2, C) operator, by means of (36) and (37):

$$\begin{aligned} L_{2,C} &= \left(\sigma_0 \cosh\frac{\eta}{2} + \mathbf{m}\cdot\boldsymbol{\sigma} \sinh\frac{\eta}{2}\right)\left(\sigma_0 \cos\frac{\delta}{2} - \mathrm{i}\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\delta}{2}\right) \\ &= \sigma_0 \cosh\frac{\eta}{2}\cos\frac{\delta}{2} + \left(\mathbf{m}\sinh\frac{\eta}{2}\cos\frac{\delta}{2} - \mathrm{i}\mathbf{n}\cosh\frac{\eta}{2}\sin\frac{\delta}{2}\right)\cdot\boldsymbol{\sigma} \\ &\quad - \mathrm{i}\mathbf{m}\cdot\mathbf{n}\sigma_0 \sinh\frac{\eta}{2}\sin\frac{\delta}{2} + (\mathbf{m}\times\mathbf{n})\cdot\boldsymbol{\sigma}\sinh\frac{\eta}{2}\sin\frac{\delta}{2} \\ &= \left(\cosh\frac{\eta}{2}\cos\frac{\delta}{2} - \mathrm{i}\mathbf{n}\cdot\mathbf{m}\sinh\frac{\eta}{2}\sin\frac{\delta}{2}\right)\sigma_0 \\ &\quad + \left[\mathbf{m}\sinh\frac{\eta}{2}\cos\frac{\delta}{2} - \mathrm{i}\mathbf{n}\cosh\frac{\eta}{2}\sin\frac{\delta}{2} + (\mathbf{m}\times\mathbf{n})\sinh\frac{\eta}{2}\sin\frac{\delta}{2}\right]\cdot\boldsymbol{\sigma}. \end{aligned} \tag{39}$$

This is the general Pauli algebraic linear expansion of a SL(2, C) operator.

Now we will analyze in what conditions the SL(2, C) operator becomes a normal one. For our polarization problem this comes to establish in what conditions the composed device is orthogonal.

There are some possibilities to approach this problem, corresponding to the various characteristic features adopted for defining the normal operators.

One of the properties of the normal operators is that their right-hand and left-hand polar decompositions coincide. In other words, the polar decomposition of a normal operator is unique and invertible (e.g. [p 103, 1]). We shall make use of this property in our approach.

Working out similarly the left-hand polar decomposition corresponding to (38), it is straightforward that the commutator of the two polar factors of the $L_{2,C}$ operator (38) is

$$[H_{\mathbf{m}}(\eta), U_{\mathbf{n}}(\delta)] = 2\mathbf{m}\times\mathbf{n}\sinh\frac{\eta}{2}\sin\frac{\delta}{2}. \tag{40}$$

Avoiding the trivial cases $\eta = 0$, $\delta = 0$, this commutator vanishes if and only if

$$\mathbf{m}\times\mathbf{n} = 0, \tag{41}$$

i.e. if the Poincaré axes of the 'operatorial modulus' and of the 'phase factor' of the operator $L_{2,C}$, given by equation (38) are parallel (coincide):

$$\mathbf{m} = \mathbf{n}. \tag{42}$$

Hence the expansion (39) of a SL(2, C) operator reduces for normal SL(2, C) operators to

$$N_{2,C} = \left(\cosh\frac{\eta}{2}\cos\frac{\delta}{2} - \mathrm{i}\sinh\frac{\eta}{2}\sin\frac{\delta}{2}\right)\sigma_0 + \left(\sinh\frac{\eta}{2}\cos\frac{\delta}{2} - \mathrm{i}\cosh\frac{\eta}{2}\sin\frac{\delta}{2}\right)\mathbf{n}\cdot\boldsymbol{\sigma}. \tag{43}$$

The last expression can be readily led to a single-exponential form

$$N_{2,C} = \mathrm{e}^{(\frac{\eta}{2}-\mathrm{i}\frac{\delta}{2})\mathbf{n}\cdot\sigma}, \tag{44}$$

where $\mathbf{n}$ is a real unit vector.

Referring to the problem of polarization devices, the Poincaré axes of dichroism and of birefringence for an orthogonal device coincide, equation (44), whereas for a non-orthogonal device they are different, equation (38).

6. Conclusions

The problem of expressing the product of two exponential operators in the form of a unique exponential operator was solved in quite general terms by the BCH formula, which gives this product as an exponential of an infinite series of commutators terms. The series cannot generally be summed explicitly, but the recursion scheme which gives the terms of the series may, in principle, be carried out to arbitrary high order.

There are some particular cases when $\mathrm{e}^A\,\mathrm{e}^B$ can be written as a closed-form single exponential e^C. Probably, the most important such a case is when A and B are 2×2 linear operators over the field of real or complex numbers, because this case refers to the mathematics supporting all the problems of 'two-level' or 'two-beams' physical systems [20].

In this paper, we have established the general solution of the problem for any 2×2 linear operator. The vectorial Pauli algebraic approach is the most economic and leads to very compact (vectorial) and symmetric results for the parameter γ and the Pauli axis $\mathbf{c}$ of the operator $C = \ln \mathrm{e}^A\,\mathrm{e}^B$, as function of the parameters μ, ν and the Pauli axes $\mathbf{m}$, $\mathbf{n}$ of the operators A and B.

We have exemplified the results for two widely encountered applications.

One of them is the classical problem of compounding two rotations. We have given the rotations by their corresponding unitary operators (hence in a spinorial representation). Equations (34) and (35) giving the elements of the resulting rotation (the rotation angle and the axis of rotation) as function of those of the composed rotations are very compact (vectorial), symmetric and physically expressive in comparison to the corresponding results given in the literature of the domain.

As a second example, we have chosen an important up-to-date problem of the polarization theory: the relationship between the elements of the birefringent and dichroic parts of a general polarization device constituted generally by a sandwich of birefringent, chiral and dichroic components. These two parts are described by the unitary and Hermitian polar factors of the operator of the composed device, respectively. In this case, unlike the previous one, the Pauli axis of the operator C is generally a complex vector. We have established also the condition under which the polarization sandwich is an orthogonal one: in this particular case the Pauli axis of the operator may be reduced to a real vector, but, in general, the parameter γ is complex (the angle of rotation is complex, i.e. the operator of the device corresponds to a rotation and a boost).

This mathematics sustains a large variety of physical problems of the so-called 'two-state' or 'two-beam' systems. Among them the very known are the spin 1/2 and the light polarization systems, for which the corresponding Hilbert state space is a bi-dimensional one. But in the last years comes to light the fact that many others physically and apparently different systems (geometric-optical [21], interferometric [22], laser [23], multilayer [24], squeeze states of light [25]) are in fact of the same kind and have a same underlying mathematics. The above-established results may be equally used in the pure-operatorial ('non-matrix') approaches to all these problems.

Acknowledgments

I am grateful to the reviewers for the valuable comments that resulted in a substantial improvement of the manuscript and for drawing my attention to the paper [7] of Dragt and Finn.

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